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Young-ping Pao

U.S. Department of Energy

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NONLINEAR EVOLUTION OF LONGITUDINAL PLASMA WAVES

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ABSTRACT

Long-time nonlinear behavior of single-mode longitudinal plasma waves is studied on the basis of Vlasov equation with Fokker-Planck collision terms. Resonant layer, trapped island, collisional sublayers and x-point neighborhoods are analyzed. Nonlinear evolution equation is obtained and wave-generated current is also evaluated.

I. INTRODUCTION

This paper presents an analytic treatment of nonlinear longitudinal plasma waves. In the linear approximation, these waves decay according to the renowned Landau formulus[1-3]. Numerous efforts have been made to examine their nonlinear effects. For multi-mode incoherent waves, a collisionless turbulent theory had been developed on the basis of selective mode coupling, giving rise to quasilinear diffusion[4-6]. At the same time, a nonlinear collisionless calculation[7,8] for single-mode waves yields a decay rate slower than the Landau result.

On a longer time scale, collisional effects play an important role in the nonlinear evolution of single-mode waves. The nonlinear collisional theory of Zakharov and Karpman[9] showed that the wave amplitude $A(t)$ decays according to $\dot{A} = -\nu\beta/A^{1/2}$, where ν is the electron collision frequency and β a positive constant.

However there appear several logical gaps in the analyses of Ref. 9. The most serious gap is the neglect of the inertial term in the non-resonant region, which leads to a direct violation of the Poisson's equation. A detailed discussion of this point is given in Section VII. The integration constant for the lowest order resonant layer solution in Ref. 9 is also in error, which directly affects the final result for the nonlinear decay rate.

These considerations motivated us to re-examine the one-dimensional problem of a single-mode electrostatic wave train of small amplitude propagating in a Maxwellian background plasma with low collision frequency ν. We shall adopt the more realistic model of full Fokker-Planck collision terms and carry out a systematic asymptotic analysis by treating the resonant and non-resonant regions separately. Our result leads to the nonlinear equation

$$\dot{A}(t) = -\nu(\gamma A + \alpha A^{1/2}) \qquad (1)$$

where the linear decay constant γ is positive and the constant α is positive for $z \leq 12$ and negative for $z \geq 13$, z being the charge number of the ions.

The linear term γA represents the damping due to collisions between the bulk thermal electrons and the ions while the nonlinear α-term arises from the collisional re-distribution of the space charge of the resonant electrons.

The main difference between the present result in eq. (1) and that of Ref. 9 is the $A^{1/2}$ dependence of the nonlinear term in (1) while Ref. 9 gives a nonlinear term with $A^{-1/2}$ dependence. The coefficient α in (1) is also completely different from that given in Ref. 9.

The basis of th present analysis is the Boltzmann-Vlasov equation with Fokker-Planck collision terms. Nonlinear and/or collisional effects become important in the velocity space resonant layer where the electron longitudinal velocity approaches the wave phase velocity. The dynamics of the resonant electrons can be dominated by nonlinearity or collisions depending on the magnitude of the parameter $\lambda = \nu/\varepsilon^3$ where ε^2 is the magnitude of the wave amplitude. For large λ, collisions prevail over nonlinearity and therefore linear theory applies and leads to Landau's result. On the other hand when λ is small, nonlinearity overtakes collisional effects for the resonant electrons and the result is qualitatively different from the linear theory.

In this paper, only the more interesting nonlinear case of small λ is considered. In this case, although nonlinearity dominates the resonant electron dynamics, collisional effects also play an indispensible role in the long time behavior of the wave train. The importance of collisions comes about in two ways. First, acting as higher order corrections, the collision terms determine the lowest order distribution function of the resonant electrons. Secondly, collisions provide the necessary transition between the trapped and circulating electrons.

The main nonlinear feature of the problem is that the deviation of the resonant electron distribution from the Maxwellian is of order ε which is proportional to the square root of the wave amplitude, while in the linear theory the deviation is of ε^2 order.

The basic equations and variables are given in the next section and solution to these equations for the non-resonant electrons are obtained in Section III.

II. EQUATIONS AND WAVE VARIABLES

Consider a single-mode wave train with wave number κ and basic frequency ω. By normalizing the distance x with $1/\kappa$, the time t with $1/\kappa v_e$, where $v_e = (T_e/m)^{1/2}$ and t_e is the electron temperature at $t = 0$. We introduce the wave variables θ and the phase shift $\psi(t)$ as

$$\theta = x - ct + \psi(t) \, , \qquad c = \omega/\kappa v_e \, , \tag{2}$$

and express the electric field E as

$$E = - \frac{\kappa m v_e^2}{e} \frac{d\phi}{d\theta} \, ,$$

where e is the (signed) electron charge. We shall also normalize the electron distribution function F by NZ/v_e^3, N being the ion number density.

In terms of these variables, the basic equations become

$$\frac{\partial F}{\partial t} + (v_x - c) \frac{\partial F}{\partial \theta} - \frac{d\phi}{d\theta} \frac{\partial F}{\partial v_x} = \nu C(F,F) \tag{3}$$

$$\frac{d^2\phi}{d\theta^2} = \left(\frac{\omega_e}{\kappa v_e}\right)^2 \left(1 - \int F \, d^3v\right) \, , \tag{4}$$

where the ions are assumed to have a constant Maxwellian distribution, and the electron plasma frequency ω_e and the electron collision frequency ν are given by

$$\omega_e^2 = 4\pi NZe^2/m ,$$

$$\nu = 2\pi NZe^4 \,\ell n\, \Lambda/\kappa m^2 v_e^4 ,$$

Λ is the Coulomb logarithm and the detailed form of the Fokker-Planck collision terms C(F,F) can be found in Ref. 10.

Because of azimuthal symmetry, the electron distribution function F has no azimuthal angle dependence and may be written as

$$F = F(v_x v_\perp, \theta, t) ,$$

where x is the direction of propagation and $v_\perp^2 = v_y^2 + v_z^2$.

Since the wave amplitude is small, the potential ϕ may be expanded as

$$\phi = \epsilon^2 \phi_o(\theta,t) + \epsilon^4 \phi_1(\theta,t) + \ldots \qquad (5)$$

where the lowest order form is $\phi_o = A(t) \cos\theta$ as anticipated from the linear theory with the provisor that A(t) varies with t.

The mathematical problem is to solve (3) and (4) for $\epsilon \ll 1$ and $\nu \ll 1$. For sake of simplicity, it is also assumed that $c \gg 1$ in conformity with many wave experiments where the wave phase velocity is much larger than the electron thermal velocity v_e.

To solve the Boltzmann equation (3), we divide the velocity space into two regions: a non-resonant region with $|v_x - c| \gg \epsilon_1$ where the linear theory provides the lowest order solution and a resonant layer with $v_x - c \sim \epsilon_1$ where the electrons must be treated nonlinearly. The scale parameter ϵ_1 will be made explicit in the next section. Once the solutions are obtained for the two regions, they can be substituted into the Poisson's equation (4) and the result may be symbolically expressed as

$$\frac{d^2\phi}{d\theta^2} = -K^2\phi + \rho_1(\theta, A, \dot{A}, \dot{\psi}, c) , \qquad (6)$$

where the K^2-term is the linear theory result from the lowest order non-resonant solution and ρ_1 is the higher order contributions from the non-resonant as well as resonant electrons.

The lowest order result of (6) is

$$\frac{d^2\phi_o}{d\theta^2} = -K^2\phi_o$$

which leads to the linear dispersion relation

$$K^2(\kappa, c) = 1$$

and our lowest order result for ϕ:

$$\phi_o = A(t) \cos\theta .$$

In the next order, we have

$$\frac{d^2\phi_1}{d\theta^2} + \phi_1 = \rho_1 .$$

It then follows that ρ_1 must satisfy two compatibility conditions, namely

$$\int_0^{2\pi} d\theta \; \rho_1(\theta,A,\dot{A},\dot{\psi},c) \sin\theta = 0 \; , \qquad (7)$$

$$\int_0^{2\pi} d\theta \; \rho_1(\theta,A,\dot{A},\dot{\psi},c) \cos\theta = 0 \; , \qquad (8)$$

and these two equations determine $\dot{A}$ and $\dot{\psi}$ and therewith the evolution of the wave train.

On the other hand, the solutions for F in the non-resonant and resonant regions can also be used in the energy equation to evaluate the change of background Maxwellian in time, which will be done in Section VII.

The details of the last two equations will be postponed until Section VI; for the moment we shall turn to the non-resonant region analysis.

III. NON-RESONANT REGION

Solutions for the non-resonant region can be obtained by iteration from the linear theory results. The distribution function F can be decomposed as

$$F = F_o(v_x, v_\perp, t) + f(v_x, v_\perp, \theta, t) , \qquad (9)$$

where F_o is the θ-averaged part and f the θ-dependent part with the θ-average

$$\langle f \rangle_\theta \equiv \frac{1}{2\pi} \int_o^{2\pi} f(v_x, v_\perp, \theta, t) d\theta = 0 , \qquad (10)$$

and f is periodic in θ with period 2π. The t-dependence in F_o and f allows for slow variation in time.

Two separate equations can be derived for F_o and f. First, equation (3) can be θ-averaged with the result

$$\frac{\partial F_o}{\partial t} - \langle \frac{d\phi}{d\theta} \frac{\partial F}{\partial v_x} \rangle_\theta = \nu C(F_o, F_o) + \nu \langle C(f,f) \rangle \qquad (11)$$

which describes the time evolution of F. Then subtracting (11) from (3) yields the following equation

$$(v_x - c) \frac{\partial f}{\partial \theta} - \frac{d\phi}{d\theta} \frac{\partial F_o}{\partial v_x} = \nu C_L(f) - \langle \frac{d\phi}{d\theta} \frac{\partial f}{\partial v_x} \rangle_\theta - \frac{\partial f}{\partial t} \qquad (12)$$

for f, where C_L is the linearized collision terms

$$C_L(f) = C_{ee}(F_o,f) + C_{ee}(f,F_o) + C_{ei}(f) \; ,$$

In the non-resonant region $|v_x-c| >> \varepsilon_1$. The last two terms in (12) will be shown to be of ν order. Therefore, if $\varepsilon_1 >> \nu$, then all the right hand side terms of (12) are much smaller than its left hand side terms. This then calls for the expansion

$$f = f_o + f_1 + \ldots$$

where $f_o \sim \varepsilon^2$, $f_1 \sim \nu\varepsilon^2$ and they satisfy, respectively,

$$(v_x-c)\,\frac{\partial f_o}{\partial\theta} - \frac{d\phi}{d\theta}\,\frac{\partial(F_o+f_o)}{\partial v_x} = 0 \; , \qquad (13)$$

$$(v_x-c)\,\frac{\partial f_1}{\partial\theta} = \nu C_L(f_o) - \frac{\partial f_o}{\partial t} - \left\langle\frac{d\phi}{d\theta}\,\frac{\partial f_o}{\partial v_x}\right\rangle_\theta \; . \qquad (14)$$

Note that in eq. (13) we have carried the higher order term $\partial f_o/\partial v_x$, just to save iteration steps. In fact, integrating (13) along the characteristic

$$Y = \frac{1}{2}\,(v_x-c)^2 + \phi$$

yields the result

$$f_o = \int^\theta \frac{d\phi}{d\theta}\,\frac{1}{(2Y-2\phi)^{1/2}}\,\frac{\partial F_o}{\partial v_x}\,d\theta \; , \qquad (15)$$

and for ϕ small, the lowest order f_o is just the linear solution

$$f_o \cong f_{oo} = \frac{\phi}{v_x - c}\frac{\partial F_o}{\partial v_x} \; . \tag{16}$$

It is clear from (15) that $f_o = f_o(v_x, v_\perp, \phi, t)$ to all orders of ϕ and consequently the last term in (14) vanishes. Then the lowest order solution of (14) becomes

$$f_1 \cong \varepsilon^2(f_{1s} \sin\theta \quad f_{1c} \cos\theta)/(v_x - c) \; , \tag{17}$$

$$f_{1s} = A(t)\nu C_L(MF_o) - \dot{A}(t)MF_o \; , \tag{18}$$

$$f_{1c} = -A(t)\dot{\psi}(t)MF_o \; , \tag{19}$$

$$MF_o = (\partial F_o/\partial v_x)/(v_x - c) \; . \tag{20}$$

Now we express F in the non-resonant region as

$$F_o = F_M + F_1 + \ldots \; , \tag{21}$$

where F_M is the Maxwellian which, in our normalized variables, becomes

$$F_M = (2\pi T)^{-3/2} \exp\,(-v^2/2T) \; ,$$

where $T = T(t)$ is the electron temperature at t normalized by the electron temperature T_e at $t = 0$.

The deviation F_1 from the Maxwellian is responsible for the generation of current drive and is therefore a quantity of interest.

Now substituting (21) into (11) yields the equation in the non-resonant region:

$$\nu C_L(F_1) = \frac{\partial F_M}{\partial t} - \langle\frac{d\phi}{d\theta}\,\frac{\partial f_1}{\partial v_x}\rangle_\theta - \nu\langle C(f_o,f_o)\rangle \ , \qquad (22)$$

which can be solved for F_1 if $\partial F_M/\partial t$ is known.

To evaluate the rate of change of F_M, we return to the whole velocity space and use the original equation (3). Multiplying (3) with $mv^2/2$ and integrating over θ and $\underset{\sim}{v}$ yield the energy equation

$$\frac{3}{2}\, m\dot{T} = -\int m v_x \langle\frac{d\phi}{d\theta}\, F\rangle_\theta d^3\underset{\sim}{v} + \frac{1}{2}\, \nu \int m v^2 C_{ei}(F_M) d\underset{\sim}{v} \ ,$$

where the last term represents the energy transfer between electrons and ions and is of the order $\nu(m/m_i)$, m_i being the ion mass. The ion-electron energy transfer can be evaluated explicitly and the final form of the energy equation then becomes

$$\begin{aligned} \frac{3}{2}\, \dot{T} &= -\int v_x \langle\frac{d\phi}{d\theta}\, F\rangle_\theta d\underset{\sim}{v} + \mu\nu(T_i - T) \ , \\ \mu &= 4z(m/m_i)/(2\pi T^3)^{1/2} \ , \end{aligned} \qquad (23)$$

where T_i is the ion temperature normalized by T_e.

It must be emphasized that the integral in (23) covers both the resonant and the non-resonant regions. We now proceed to the resonant layer analysis where nonlinearity plays an important role.

IV. RESONANT LAYER

In the resonant layer, we introduce the new variable u and write

$$v_x - c = \varepsilon_1 u$$

where ε_1 is the length scale in the layer. The Vlasov equation (3) can then be written as

$$\varepsilon_1 u \frac{\partial F}{\partial \theta} + \frac{\varepsilon^2 A \sin\theta}{\varepsilon_1} \frac{\partial F}{\partial u} = \nu C_L(F) - \frac{\partial F}{\partial t} , \qquad (24)$$

and the linearized collision terms $C_L(F)$ takes the simplified form

$$C_L(F) \doteq (1+z) \frac{\partial}{\partial \mu} \left[\frac{(1-\mu^2)}{v^3} \frac{\partial F}{\partial \mu}\right] + \frac{2}{v^2} \frac{\partial}{\partial v} \left(\frac{1}{v} \frac{\partial F}{\partial v} + F\right) \qquad (25)$$

in terms of the pitch angle variable $\mu = v_x/v$.

For resonant electrons, most of the collisions are with the bulk thermal electrons and ions. Since the longitudinal velocity of the resonant electron is close to c which is much larger than the thermal speed of the bulk electrons and ions, the Fokker-Planck collision terms can be simplified to yield the result in (25). In terms of the variables u and $v_\perp$, eq. (25) can be further simplified as

$$C_L(F) \doteq \frac{\nu}{\varepsilon_1^2 c^3} C_R(F)$$

where the resonant-layer collision terms $C_R(F)$ are

$$C_R(F) = \frac{\nu_o}{\nu}\frac{\partial^2 F}{\partial u^2} + \varepsilon_1 c L_1 \frac{\partial F}{\partial u} + \varepsilon^2 c^2 L_2 F \ ,$$

$$\nu_o = \nu[(1+z)v_\perp^2 + 2T] \ , \tag{26}$$

$$L_1 = -2[z + (1+z)D_\perp] \ ,$$

$$L_2 = (1+z)v_\perp^{-2}D_\perp^2 \ , \qquad D_\perp = v_\perp \frac{\partial}{\partial v_\perp} \ ,$$

and $v_\perp$ is considered to be of order unity.

Using $\varepsilon_1 c$ as a small parameter, we obtain the lowest order result of (24) as

$$u\frac{\partial F}{\partial \theta} + \frac{\varepsilon^2 A \sin\theta}{\varepsilon_1^2}\frac{\partial F}{\partial u} = \frac{\nu_o}{\varepsilon_1^3 c^3}\frac{\partial^2 F}{\partial u^2} \ , \tag{27}$$

where we have neglected the inertial term in (24) by assuming $\partial/\partial t << \nu/\varepsilon_1^2 c^3$; i.e., we look for long-time behavior of the wave propagation and the neglect of the inertial term will be justified a posteriori. In the opposite limit of $\partial/\partial t >> \nu/\varepsilon_1^2 c^3$, the collision terms can be neglected in (24) and the problem reduced to the collisionless nonlinear case that has already been treated by O'Neil.[7]

The nature of eq. (27) depends on the parameter $\lambda = \nu/\varepsilon^3 c^3$. For large λ, collisions dominate nonlinearity while small λ means that nonlinearity prevails over collisions. We discuss these two cases separately.

(i) $\lambda >> 1$. We choose $\varepsilon_1 = \nu^{1/3}c$ and rewrite (27) as

$$[(1+z)v_{\perp}^2 + 2T]\frac{\partial^2 F}{\partial u^2} - u\frac{\partial F}{\partial \theta} = \lambda^{-2/3}A\sin\theta\frac{\partial F}{\partial u} ,$$

which can be solved by expanding in the small parameter $\lambda^{-2/3}$ and the lowest order solution can be constructed from Airy functions. The final result, obtained by matching the resonant layer solutions to the non-resonant solutions, agrees with the linear theory of Landau damping. Although collisions dominate over nonlinearity, we must bear in mind that $\nu << 1$; i.e., collision frequency is small in comparison with kc. It is not surprising that this should lead to the result of collisionless linear theory.

(ii) $\lambda << 1$. In this case, we choose $\varepsilon_1 = \varepsilon$ and rewrite (27) as

$$u\frac{\partial F}{\partial \theta} + A\sin\theta\frac{\partial F}{\partial u} = \lambda_o\frac{\partial^2 F}{\partial u^2} , \qquad (28)$$

where we have $\lambda_o = \lambda\nu_o/\nu << 1$.

Our main interest lies in Case (ii) where nonlinearity plays a dominant role. For our purpose, the higher order collision terms in (26) are also important and we therefore shall retain these terms in (28) and consider

$$u\frac{\partial F}{\partial \theta} + A\sin\theta\frac{\partial F}{\partial u} = \lambda C_R(F) , \qquad (29)$$

where $C_R(F)$ contains all the terms in (26).

We can now solve (29) by expanding for small λ:

$$F = \hat{F}_0 + \lambda \hat{F}_1 + \ldots$$

and the lowest order equation for $\hat{F}_0$ is

$$u \left(\frac{\partial \hat{F}_0}{\partial \theta}\right)_\xi = 0 ,$$

where the characteristic variable ξ is just the one-dimensional energy integral

$$\xi = \frac{1}{2} u^2 + A \cos \theta \quad , \qquad (30)$$

and the solution is

$$\hat{F}_0 = \hat{F}_0(\xi) .$$

The constant ξ contours in the (u,θ)-plane form an island structure where the separatrix at $\xi = A$ separates the circulating electrons for $\xi > A$ from the trapped electrons for $\xi < A$; see Fig. 1. The solutions take different form for the two classes of electrons as shown in the following.

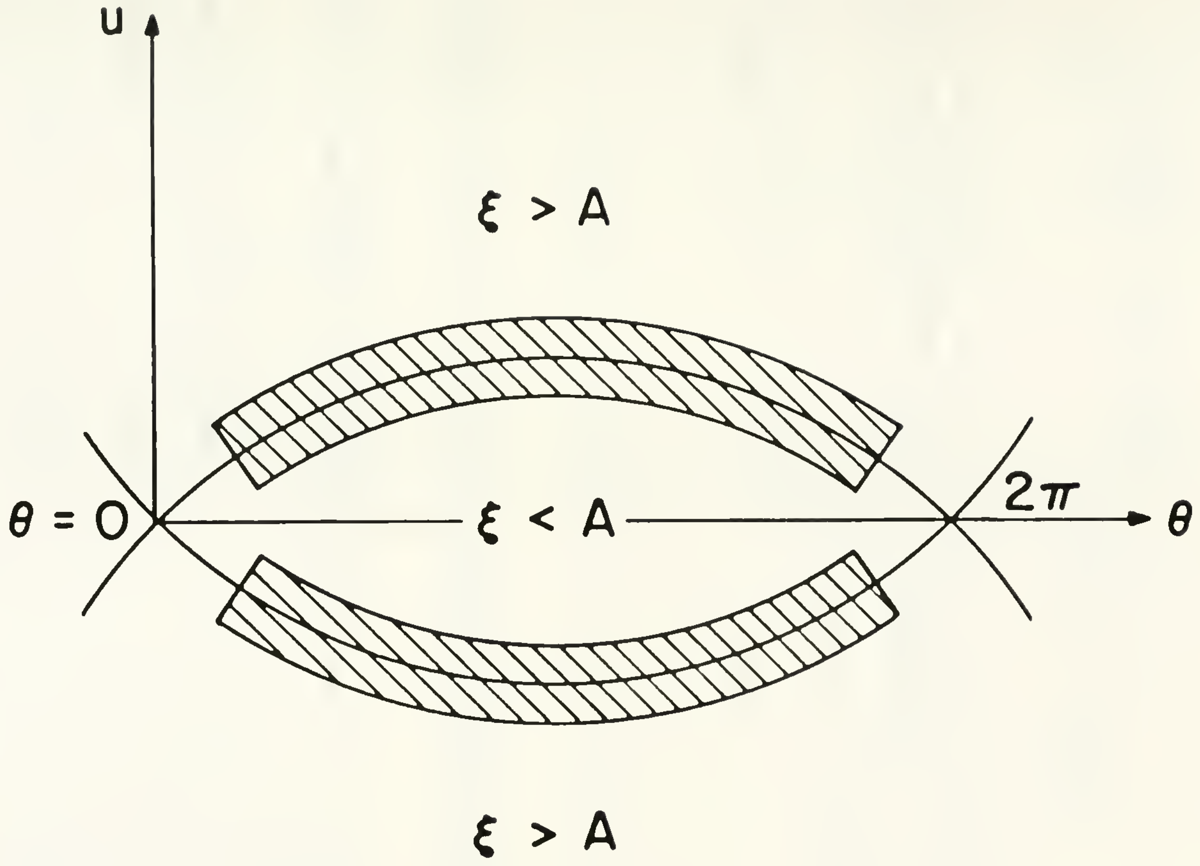

FIGURE 1

Trapped electron island. Shaded strips are collisional sub-layers.

IV.A. Solutions for Circulating Electrons

The first order terms in (29) satisfy the equation

$$\frac{\partial \hat{F}_1(\xi,\theta)}{\partial\theta} = \frac{\lambda}{u} C_R(\hat{F}_o) \; , \tag{31}$$

and since $\hat{F}_1$ must be periodic in θ, we can integrate (31) over θ for fixed ξ to obtain

$$\langle C_R(\hat{F}_o)/u \rangle = 0 \; , \tag{32}$$

where the bracket denotes the average

$$\langle F \rangle = \frac{1}{2\pi} \int_o^{2\pi} F(\xi,\theta)d\theta \; , \qquad \xi > A \tag{33}$$

with ξ kept constant; this is different from the θ-average in (10) which keeps v_x fixed.

We now solve (32) for $\hat{F}_o$ by substituting (26) into (32) and expanding in the small parameter $\varepsilon_1 c$, where now $\varepsilon_1 = \varepsilon$. If we express $\hat{F}_o$ as

$$\hat{F}_o(\xi) = F_c + \varepsilon c G(\xi) + \varepsilon^2 c^2 H(\xi) + \dots \; , \tag{34}$$

where F_c is the Maxwellian distribution at $v_x = c$, viz.,

$$F_c = (2\pi T)^{-3/2} \exp\,[-(v_\perp^2+c^2)/2T] \; .$$

Then substitution of (26) into (32), up to $\varepsilon^2 c^2$ order, yields

$$\langle \frac{1}{u} \frac{\partial^2 G_\pm}{\partial u^2} \rangle = 0 \ , \qquad (35)$$

$$\frac{\nu_o}{\nu} \langle \frac{1}{u} \frac{\partial^2 H_\pm}{\partial u^2} \rangle + \langle \frac{1}{u} L_1 \frac{\partial g_\pm}{\partial u} \rangle + \langle \frac{1}{u} L_2 F_c \rangle = 0 \ , \qquad (36)$$

where we have used the + and - subscripts to denote the directions of the electron motion with $u > 0$ and $u < 0$, respectively.

As $\xi \to \infty$, or $u \to \pm\infty$, $\hat{F}_o$ must match the solution in the non-resonant region for $v_x \to c$, namely

$$F_M = F_c \left[1 - \frac{\epsilon c}{T} u + \frac{\epsilon^2 c^2}{2T^2} u^2 + \dots \right] .$$

This then shows that the boundary conditions for G and H are:

$$G_\pm(\xi) \to \mp \sqrt{2\xi}\, F_c/T \ , \qquad (37)$$

$$H_\pm(\xi) \to \xi F_c/T^2 \ , \qquad (38)$$

for $u \to \pm\infty$.

The solution for (35) subject to the boundary condition (37) is given by

$$G_\pm(\xi) = \mp(F_c/T)\left[\sqrt{2\xi} - \chi(\xi)\right] \ , \qquad (39)$$

$$\chi(\xi) = \int_\xi^\infty \left[\frac{1}{\langle|u|\rangle} - \frac{1}{\sqrt{2\eta}}\right] d\eta \ , \qquad (40)$$

$$|u| = (2\eta - 2A \cos \theta)^{1/2} \ .$$

To solve (36), we use $\partial/\partial u = u\partial/\partial\xi$ and rewrite eq. (36) as

$$[\frac{\nu_o}{\nu} \langle u\rangle H_{\pm}' + L_1 G_{\pm} + \langle u\rangle L_2 F_c]' = 0 , \tag{41}$$

where the prime denotes differentiation with respect to ξ.

We now integrate (41) and set the integration constant to zero on account of (38) and the final result becomes

$$H_{\pm}'(\xi) = F_c T^{-2}[1 + P(\xi)Q(v_{\perp}^2/2T)/\langle|u|\rangle] , \tag{42}$$

$$P(\xi) = \sqrt{2\xi} - \langle|u|\rangle - \chi(\xi) , \tag{43}$$

$$Q(x) = (2x-\sigma z)/(x+\sigma) , \qquad \sigma = \frac{1}{(1+z)} , \tag{44}$$

where $P(\xi) \to 0$ as $u \to \pm\infty$ so that (42) satisfies (38) and $\chi(\xi)$ is given in (40).

IV.B. Solutions for Trapped Electrons

For trapped electrons with $\xi < A$, the ξ contour is bounded in θ between θ_ℓ to the left of $\theta = \pi$ and θ_r to the right of $\theta = \pi$, θ_ℓ and θ_r being the two roots of $\cos\theta = \xi/A$.

Similar to the circulating electron solutions, the lowest order (in λ) solution for the trapped electrons has the form

$$\hat{F}_o = F_o(\xi)$$

and the λ-order equation is again (31). Now we cannot use the periodicity argument. Instead, we invoke the continuity of $\hat{F}_1$ as $u \to \pm 0$ and the result becomes

$$\langle C_*(\hat{F}_o)/|u|\rangle = 0 \ , \tag{45}$$

where C_* is

$$C_*(F) = \frac{\nu_o}{\nu}\,|u|\,\frac{\partial}{\partial\xi}\left(|u|\,\frac{\partial F}{\partial\xi}\right) + \varepsilon c L_1\{u|\,\frac{\partial F}{\partial\xi} + \varepsilon^2 c^2 L_2 F \ , \tag{46}$$

and the bracket in (45) is defined as

$$\langle F\rangle = \frac{1}{2\pi}\int_{\theta_\ell}^{\theta_r} F(\xi,\theta)d\theta \ , \tag{47}$$

for $\xi < A$. In other words, for $\xi > A$ the bracket $\langle F\rangle$ is defined in (33) and for $\xi < A$ it is defined in (47).

Again, we shall solve (45) by expanding in εc:

$$\hat{F}_o = F_c + \varepsilon c G_*(\xi) + \varepsilon^2 c^2 H_*(\xi) + \ldots \ , \tag{48}$$

which, upon substituting into (45), leads to the result

$$[\langle|u|\rangle G_*'(\xi)]' = 0 \ , \tag{45}$$

$$[\frac{\nu_o}{\nu}\langle|u|\rangle H_*' + L_1 G_* + \langle|u|\rangle L_2 F_c]' = 0 \ . \tag{46}$$

The solution for (45) is

$$G_*(\xi) = c_1 + c_2 \int_A^\xi d\xi_1/\langle|u|\rangle \qquad (47)$$

which must match (39) as $\xi \to A$ for $u > 0$ and $u < 0$. This is impossible since $G_\pm$ in (39) have different signs. The way out of this dilemma is to allow for a singular layer near the separatrix at $\xi = A$ where the collision terms cannot be treated as small. These collisional sublayers will be examined in the next section. One of the results from the collisional sublayer analysis shows that the constants in (47) must vanish, i.e., $c_1 = c_2 = 0$, and consequently

$$G_*(\xi) \equiv 0 \ .$$

We can now integrate (46) with the result

$$H_*'(\xi) = F_c T^{-2}[1 - Q(v_\perp^2/2T)] \ , \qquad (48)$$

where Q and σ is given in (44). In deriving (48), we have used the boundary condition that H_*' is regular at the island center $\xi = -A$.

It is also noted that (48) does not match (42) at the separatrix so that H_*' must also undergo a rapid transition through the collisional sublayers. However, $H_*(\xi)$ can match $H_\pm(\xi)$ at $\xi = A$ by adjusting the constant of integration; i.e., $H_\pm(\xi)$ and $H_*(\xi)$ are continuous at $\xi = A$.

Finally, we note that we have neglected the $\partial F/\partial t$ term in (24) in obtaining the resonant layer solutions. This term may be expressed as

$$\frac{\partial F}{\partial t} = \dot{A}\frac{\partial F}{\partial A} + \dot{\psi}\frac{\partial F}{\partial \theta} + \dot{T}\frac{\partial F}{\partial t} \ .$$

The neglect of the $\dot{A}$ and $\dot{T}$ terms can be justified since, as we shall see later, $\dot{A} \sim \nu$ and $\dot{T} \sim \nu\varepsilon^2$ which is much smaller than the λC_R term in (24). On the other hand it will be shown that $\dot{\psi} \sim \varepsilon^4 c^2$ and therefore the $\dot{\psi}$ term can be neglected in comparison with the lowest order term $u\partial F/\partial\theta$. Furthermore, retention of the $\dot{\psi}$ term as a higher order term in (24) or (31) will not affect the results in (32) or (45) because

$$<\dot{\psi} \frac{\partial \hat{F}_o}{\partial\theta} \frac{1}{u}> = \dot{\psi}\hat{F}_o' <\frac{\partial\xi}{\partial\theta} \frac{1}{u}> = 0$$

on account of the antisymmetry in θ and this justifies the neglect of $\partial F/\partial t$ in the resonant layer analysis.

V. COLLISIONAL SUBLAYERS AND x-POINT

We now proceed to analyze the collisional sublayers and to substantiate our previous claim that the constants c_1 and c_2 in (47) vanish. Our analysis follows the method developed in Ref. 11 for hydrodynamic stability problems; for a somewhat different version of this method see Ref. 12.

There are two branches of the sublayer at $\xi = A$, an upper branch for $u > 0$ and a lower one for $u < 0$. In the sublayer, $\xi \cong A$ and we therefore introduce a new variable η through

$$\xi = A + \lambda_o^{1/2}\eta$$

$$\hat{F}_o = F_c + g_o(\eta,\theta) + \lambda_o^{1/2} g_1(\eta,\theta) + \ldots .$$

The velocity variable u can be expanded as

$$u \cong \pm 2A^{1/2}(\sin \frac{1}{2}\,\theta)\left(1 + \lambda_o^{1/2}\eta/2A \sin^2 \frac{1}{2}\,\theta\right) , \tag{49}$$

provided that $\theta >> \lambda_o^{1/4}$ and $2\pi - \theta >> \lambda_o^{1/4}$. Near the x-points where $\theta \sim \lambda_o^{1/4}$ or $2\pi-\theta \sim \lambda_o^{1/4}$, a different analysis is needed.

Substituting the expansion for $\hat{F}_o$ and (49) into (29) and using the dominant term in C_R, we obtain

$$\frac{\partial g_o}{\partial \theta_\pm} = \frac{\partial^2 g_o}{\partial \eta^2} , \tag{50}$$

$$\frac{\partial g_1}{\partial \theta_\pm} - \frac{\partial^2 g_1}{\partial \eta^2} = \frac{1}{4A \sin^2 \frac{1}{2}\theta} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial g_0}{\partial \eta}\right) , \tag{51}$$

where the $\theta_\pm$ variables are

$$\theta_\pm = 4A^{1/2}(1 \mp \cos \tfrac{1}{2}\theta) , \tag{52}$$

and the plus and minus subscripts denote $u > 0$ and $u < 0$, respectively.

As $\eta \to \pm\infty$, g_0 and g_1 must match (39) and (41). For this purpose, we rewrite g_0 as

$$g_0 = \frac{1}{2} G_\pm(A)[1 + \mathrm{erf}(\eta/z\theta_\pm^{1/2}] + \frac{1}{2} c_1[1 - \mathrm{erf}(\eta/2\theta_\pm^{1/2}] + \Upsilon(\eta,\theta_\pm) , \tag{53}$$

where the error function approaches ±1 as $\eta \to \pm\infty$.

Using a generic variable Ξ for θ_+ and θ_-, we have

$$\frac{\partial \Upsilon(\eta,\Xi)}{\partial \Xi} = \frac{\partial^2 \Upsilon(\eta,\Xi)}{\partial \eta^2} , \tag{54}$$

and Υ tends to zero as $\eta \to \pm\infty$.

As $\theta \to 0$ and $\theta \to 2\pi$, the solution in (53) must match the x-point solutions which will be shown later to be a function of η only. This matching is then equivalent to matching the expression in (53) to itself at $\theta = 0$ and $\theta =$ yielding four conditions on $\Upsilon_\pm$:

For $\eta > 0$,

$$\Upsilon_\pm(\eta,0) + G_\pm(A) = \frac{1}{2} G_\pm(A)E_0 + \frac{1}{2} c_1E_1 + \Upsilon_\pm(\eta,\Xi_0) \ , \qquad (55)$$

For $\eta < 0$

$$\Upsilon_\pm(\eta,0) + c_1 = \frac{1}{2} G_\mp(A)E_0 + \frac{1}{2} c_1E_1 + \Upsilon_\mp(\eta,\Xi_0) \ , \qquad (56)$$

where $G_\pm(A)$ is given in (39) and E_1 and E_2 are given by

$$\begin{aligned} E_n &= E_n(\eta) = 1 + (-1)^n \mathrm{erf}(\eta/z\,\Xi_0^{1/2}) \ , \\ \Xi_0 &= 8A^{1/2} = \theta_+(\theta = 2\pi) = \theta_-(\theta = 0) \ . \end{aligned} \qquad (57)$$

The derivation of (57) and (58) is similar to that used in Ref. 11. Equations (55) and (56) can be converted into a simpler form in terms of

$$S(\eta,\Xi) = \Upsilon_+(\eta,\Xi) + \Upsilon_-(\eta,\Xi) \ ,$$

$$D(\eta,\Xi) = \Upsilon_+(\eta,\Xi) - \Upsilon_-(\eta,\Xi) \ ,$$

and the resultant equations become

$$S(\eta,0) - S(\eta,\Xi_0) = c_1(\mathrm{sgn}\ \eta)E_1(|\eta|) \ , \qquad (58)$$

$$D(\eta,0) - (\mathrm{sgn}\ \eta)D(\eta,\Xi_0) = -G_\pm(A)E_1(|\eta|) \ . \qquad (59)$$

Now we can show that $c_1 = 0$ by using the transform

$$\hat{S}(\zeta,\Xi) = \int_{-\infty}^{\infty} S(\eta,\Xi)\ \exp\ (i\eta\zeta)d\eta$$

which exists and is continuous since $S \to 0$ as $\eta \to \pm\infty$. Since S satisfies (54), the transform $\hat{S}$ has the property that

$$\hat{S}(\zeta,\Xi) = \hat{S}(\zeta,0)\exp(-\zeta^2\Xi) , \qquad (60)$$

while the transform version of (58) is

$$\hat{S}(\zeta,0)[1 - \exp(-\zeta^2\Xi_0)] = c_1\bar{E}_1(\zeta) , \qquad (61)$$

where $\bar{E}_1$ is the Fourier transform of $(\operatorname{sgn}\eta)E_1(|\eta|)$.

In the limit of $\zeta \to 0$, eq. (61) becomes

$$\hat{S}(0,0)\zeta^2\Xi_0 = i\, c_1 \int |\eta|E_1(|\eta|)d\eta$$

and we readily conclude from the continuity of $\hat{S}$ that

$$c_1 = 0$$

as claimed in the last section. A direct consequence of this is that $\hat{S}(\zeta,0) \equiv 0$ and therefore

$$S(\eta,\Xi) \equiv 0 . \qquad (62)$$

Equation (59) can be solved by Wiener-Hopf method, provided the singularity of the half-range transforms at $\zeta = 0$ is taken care of. The solution, together with (62) yield the solutions $Y_\pm(\eta,\Xi)$, but these will not be needed for our calculation.

Similarly, we can use the same procedure for g_1 and prove that

$$c_2 = 0 \; .$$

It is useful to observe that

$$g_i(\eta,\theta,\ u > 0) = -g_i(\eta,2\pi-\theta,\ u < 0) \; , \qquad i=0,1, \qquad (63)$$

which follows from (50), (51) and the condition that g_0 and g_1 must match (39) and (41). Equation (63) expresses the skew-symmetry for the sublayer solutions.

Finally, we consider the neighborhoods near the x-points at $u = 0$ and $\theta = 0,\ 2\pi$. In these neighborhoods, $\theta \sim \lambda^{1/4}$ (or $2\pi-\theta \sim \lambda^{1/4}$) and $u \sim \lambda^{1/4}$, and consequently the lowest order x-point equation is

$$u \frac{\partial F}{\partial \theta} - A \sin\theta \frac{\partial F}{\partial u} = 0 \; ,$$

which shows F is a function of ξ (or η) only, as claimed. Higher order solutions can be constructed but we shall not go into further details since they are tedious and not needed for our purpose. Suffice it to say that the sublayers and the x-point neighborhoods provide a smooth (though rapid) transition for the solutions of the resonant electrons. What is needed for our later calculation are the conclusions $c_1 = c_2 = 0$ and the skew-symmetry in (63).

VI. SPACE CHARGE AND NONLINEAR EVOLUTION

To determine $\dot{A}$ and $\dot{\psi}$ from (7) and (8), it is necessary to calculate the space charge in the Poisson's equation. We first consider the sin θ component ρ_s of the space charge and split it into a non-resonant part ρ_{sn} and a resonant part ρ_{sr} as

$$\rho_s = \int d\theta d\underset{\sim}{v} F \sin\theta = \rho_{sn} + \rho_{sr} , \tag{64}$$

$$\rho_{sn} = \int_0^{2\pi} d\theta \sin\theta \int_n d\underset{\sim}{v} F , \tag{65}$$

$$\rho_{sr} = \int_0^{2\pi} d\theta \sin\theta \int_r d\underset{\sim}{v} F , \tag{66}$$

where the ρ_{sn} integral for non-resonant electrons with subscript n is over $|v_x - c| > \delta$ and the ρ_{sr} integral for resonant electrons with subscript r is over $|v_x - c| \leq \delta$, δ being a parameter satisfying $\delta \ll 1$ and $\delta \gg \varepsilon$.

For the non-resonant integral in (67), we observe that only f_1 in (17) has a non-vanishing contribution while the integrals over F and f_o vanish due to symmetry in θ. Substituting (17) and (18) into (65) and expanding for $c \gg 1$, we have

$$\rho_{sn} \simeq 2\varepsilon^2 c^{-3}(\dot{A} + \gamma\nu_e A) + \delta\text{-terms} , \tag{67}$$

$$\gamma = 4/3(2\pi)^{1/2} , \qquad \nu_e = \nu T^{-3/2} , \tag{68}$$

where the δ-terms are terms that depend on δ. Note that the first two terms on the right hand side of (67) respresent the contribution of the thermal electrons and ions while the δ-terms come from the boundary at $v_x = c + \delta$ and $v_x = c - \delta$. These δ-terms must cancel the boundary terms in the resonant part ρ_{sr}.

The simplest way to derive (67) is to substitute (18) into (65), expand for large c and ignore the gap $c-\delta \leq v_x \leq c+\delta$. In a more rigorous way of deriving (67), we should subtract the singularity of the integrand in (67) as $v_x \to c$ and then take the limit of large c, but this is very tedious and we shall not dwell on it here.

Now to calculate (66), we rewrite ρ_{sr} as

$$\rho_{sr} = 2\pi \int_0^\infty v_\perp dv_\perp \bar{\rho}_{sr} \,, \tag{69}$$

$$\bar{\rho}_{sr} = \varepsilon \int_0^{2\pi} d\theta \sin\theta \int_{-\Delta}^{\Delta} du\, F \,, \tag{70}$$

where $\delta = \delta/\varepsilon >> 1$, and we have used $v_x - c = \varepsilon u$.

We now change the variables (u,θ) to (ξ,θ) where ξ is defined in (30) and use α to denote the sign of u; i.e., $\alpha = \pm 1$ for $u > 0$ and $u < 0$, respectively. Then (70) becomes

$$\bar{\rho}_{sr} = \varepsilon \sum \int_0^{2\pi} d\theta \sin\theta \int_{\xi_*}^{\Delta_*} d\xi\, F(\xi,\theta;\alpha)/U \tag{71}$$

$$\Delta_* = \frac{1}{2}\Delta^2 + \xi_* \,, \qquad \xi_* = A\cos\theta \,, \qquad U = |u| \,, \tag{72}$$

where the argument α in F indicates that F is different for $u > 0$ $(\alpha = 1)$ and $u < 0$ $(\alpha = -1)$, and the summation in (71) is over $\alpha = 1$ and $\alpha = -1$.

Noting that $A \sin\theta/U = \partial U/\partial\theta$, we can integrate (71) by parts in θ and write the result as

$$\bar{\rho}_{sr} = -\varepsilon A^{-1} \sum \int d\theta \int d\xi\, U \frac{\partial F}{\partial\theta} \,. \tag{73}$$

The θ-derivative in (73) is just the left hand side term of (29) except for the different signs of U and u. Consequently using $\alpha = \text{sgn}\,(u)$, we may write

$$\bar{\rho}_{sr} = -\lambda\varepsilon A^{-1} \sum \alpha \int_{o}^{2\pi} d\theta \int_{\xi_*}^{\Delta_*} d\xi\, C_R(F) \tag{74}$$

and if we substitute C_R in (29) into (74) and use the ξ variable, eq. (74) becomes

$$\bar{\rho}_{sr} = -\lambda\varepsilon A^{-1} \sum \int_{o}^{2\pi} d\theta \int_{\xi_*}^{\Delta_*} \left[\frac{\nu_o\alpha}{\nu} \frac{\partial}{\partial\xi} \left(u^2 \frac{\partial F}{\partial\xi} - F\right) + \varepsilon c L_1 U \frac{\partial F}{\partial\xi} + \alpha\varepsilon^2 c^2 L_2 F\right] . \tag{75}$$

The first term in the bracket of (75) is a total derivative and can be immediately integrated. At the upper limit, F approaches the Maxwellian and therefore the contribution vanishes on account of the sign in α except for higher order δ-terms. At the lower limit $\xi = \xi_*$, $u = 0$ and therefore the u^2-term vanishes while the F terms cancel out since $\sum \alpha F(\xi_*\alpha) = F(\xi_*,1) - F(\xi_*,-1) = 0$

as F must be continuous at $u = 0$.

We now substitute (75) into (74) for ρ_{sr}. The L_2F term vanishes upon integratig over $v_\perp$, and consequently

$$\rho_{sr} = -2\pi\lambda\varepsilon^2 c A^{-1} \sum \int_{o}^{\infty} v_\perp dv_\perp \int_{o}^{2\pi} d\theta \int_{\xi_*}^{\Delta_*} d\xi L_1 U \hat{F}_o' \ \delta\text{-terms}, \tag{76}$$

where the solution $\hat{F}_o$ has the form in (34). The F_c term in (34) makes no contribution to the integral in (76) since it is independent of ξ. contribution of the G term also vanishes since $G_\pm$ is anti-symmetric in the resonant layer outside the collisional sublayer, while inside the sublayer the solution is skew-symmetric; see (63). The final result is then

$$\rho_{sr} = -2\pi\lambda\varepsilon^4 c^3 A^{-1} \sum \int_0^\infty v_\perp dv_\perp \; L_1 \int_0^{2\pi} d\theta \int_{\xi_*}^{\Delta_*} d\xi U H' + \delta\text{-terms} , \qquad (77)$$

Two more simplifications are in order. First, as H' has a finite jump across the collisional sublayer, its value is bounded inside the sublayer. Since the sublayer width is of $\lambda^{1/2}$ order, the sublayer contribution to (77) can be neglected. We can then use (42) for H in $\xi > A$ and (48) for H in $\xi < A$ without paying attention to the sublayer solutions. Secondly, we observe that the expressions in (42) and (48) are independent of the sign of u, and therefore we only need to calculate the integral in (79) for $u > 0$ and double the obtained result. If we take all these steps and change the order of integrations in (77), we obtain

$$\rho_{sr} = -8\pi^2\lambda\varepsilon^4 c^3 A^{-1} \int_0^\infty v_\perp dv_\perp L_1 \left[\int_A^{\Delta_*} d\xi \langle U\rangle H'_+ + \int_{-A}^{A} d\xi \langle U\rangle H'_* \right] + \delta\text{-terms} , \qquad (78)$$

where H'_+ is given in (42) and H'_* in (48), and L_1 is defined immediately after (26). We then substitute (42) and (48) into (78), cast the boundary terms for $\xi = \Delta_*$ aside into the δ-terms and carry out the integrations. The final result for ρ_{sr} is

$$\rho_{sr} = 8\pi^{1/2}\beta_* \nu_e T^{-1} \varepsilon A^{1/2} q(\sigma) \exp\left(-c^2/2T\right) + \delta\text{-terms} , \qquad (79)$$

where the constant β_* is given by

$$\beta_* = \frac{4}{3} + \int_1^\infty d\eta(\eta-1)[\eta^{1/2} - V(\eta)]/\eta^{1/2}V(\eta) \ , \tag{80}$$

$$V(\eta) = \int_0^{2\pi} (\eta - \cos\theta)^{1/2} d\theta/2\pi \ , \tag{81}$$

the function $q(\sigma)$ is defined as

$$q(\sigma) = 2 - \int_0^\infty [(\sigma+1)/(\sigma+x)] \exp(-x)dx \ , \tag{82}$$

and $\sigma = 1/(1+z)$.

We can now substitute (67) and (79) into (64) and set $\rho_s = 0$ where the δ-terms in (67) and (79) should cancel out. This then gives us the evolution equation

$$\rho_s = 2\varepsilon^2 c^{-3}\left(\dot{A} + \gamma\nu_e A + \beta\nu_e A^{1/2}/\varepsilon\right) = 0 \ , \tag{83}$$

or

$$\dot{A} = -\gamma\nu_e A - \beta\nu_e A^{1/2}/\varepsilon \ , \tag{84}$$

where β is

$$\beta = 4\pi^{1/2}\beta_*(c^3/T)q(\sigma) \exp(-c^2/2T) \ , \tag{85}$$

where γ and β_* are positive constants given in (68) and (80), respectively, and ν_e is also given in (68).

The function $q(\sigma)$ can be expressed in terms of the exponential integrals and has the property[13] that $q > 0$ for $z \leq 11$ and $q < 0$ for $z \geq 12$. This shows that while the isotropic collision terms in (25) involving the v-derivatives are stabilizing, the pitch-angle collision terms involving μ-derivatives are destabilizing the small-$v_\perp$ electrons.

Had we neglected the isotropic collision terms, we would have an infinite nonlinear growth rate and the problem would become spurious.

Using similar procedures we can evaluate the cosine-component of the space charge. The lowest order result gives the linear dispersion relation, while the higher order terms determine $\dot{\psi}(t)$ according to eq. (8). We shall skip the details of these calculations and simply state that

$$\dot{\psi} \sim \varepsilon^6 A^3 / c^4 \ .$$

VII. TEMPERATURE CHANGE AND CURRENT DRIVE

We now calculate the change of temperature in the background Maxwellian. In doing this, we shall also be able to demonstrate the cause of the discrepancy between the present result and that obtained in Ref. 9.

Let us consider eq. (23) and recall that the θ-derivative of ϕ is simply $-\varepsilon^2 A \sin\theta$ to the lowest order in ε. For our purpose it is convenient to rewrite eq. (23) as

$$\frac{3}{2}\dot{T} = -Ac\rho_s + \frac{A}{2\pi}\int d\theta d\underset{\sim}{v}(v_x-c)F \sin\theta + \mu\nu(T_i-T) \ , \qquad (86)$$

where the sine component ρ_s of the space charge is given in (64). Note that $\rho_s = 0$ according to eq. (7).

The integral in (86) can be evaluated using the non-resonant solution in (17) and the resonant solution and the dominant contribution is from the $\dot{A}$ term in (18), namely

$$\int d\theta d\underset{\sim}{v}(v_x-c)F \sin\theta \simeq 2\pi\dot{A}/c^2 \ , \qquad (87)$$

which, upon substituting into (86), yields

$$\frac{3}{2}\dot{T} = -\varepsilon^4 A\dot{A}/c^2 + \mu\nu(T_i-T) \ . \qquad (88)$$

It can be shown by using the dispersion relation that the $\varepsilon^4 A\dot{A}/c^2$ term is just the rate of change of the wave energy $E^2/8\pi$ averaged over θ. Therefore eq. (88) simply expresses the conservation of the sum of the electron thermal energy and the wave energy except for the energy transfer between electrons and ions. It also shows that $\dot{T}$ is much smaller than $\dot{A}$ or ν.

At this juncture, it is of interest to examine the result of Zakharov and Karpman.[9] They used the wave energy $W = \langle E^2 \rangle / 8\pi$ and evaluate its rate of change according to

$$\dot{W} = - \langle Ej \rangle \ , \tag{89}$$

where j is the current in the x direction.

In terms of our non-dimensional variables, (89) can be written as

$$\varepsilon^2 A\dot{A}/c^2 = -cA\rho_s - (A/2\pi) \int d\theta d\underset{\sim}{v}(v_x - c)F \sin\theta \ , \tag{90}$$

which, upon substituting (83) and (87), becomes

$$\dot{A} = -2(\dot{A} + \gamma\nu_e A + \beta\nu_e A^{1/2}/\varepsilon) + \dot{A} \ , \tag{91}$$

where the last term is due to the integral in (90) and the bracketed term comes from ρ_s.

The procedure used in Ref. 9 is to ignore the integral in (90) on the ground that v_x-c is small in the nonlinear layer. It further neglects the first two non-resonant terms in the right hand side of (91). If we do this, we would end up with a result that differs by a factor of two with the β-term in (84). The proper way of determining $\dot{A}$ is, of course, to use $\rho_s = 0$ as in (83). If the last term of (91) is retained, Zakharov and Karpman would obtain the correct result. In any case, since the last term in (90) is equal to the $\dot{W}$ term on the left hand side, the omission of this term is certainly erroneous.

We further note that the decay term in Ref. 9 is of $A^{-1/2}$ order instead of $A^{1/2}$ as obtained in (84). As far as we can determine, this is related to the first term in the bracket of (75). If this term does not vanish, it would lead to an $A^{1/2}$ result. However we have shown in the last section that it must vanish. There are other errors in Ref. 9, such as the sign of ϕ in their characteristic variable ε in eq. (17) of Ref. 9 and the integration constant in eq. (24) of Ref. 9. These make a complete comparison of the present results with those of Ref. 9 somewhat complicated and we shall not discuss the details further.

Finally we wish to calculate the d.c. current j generated by the longitudinal wave

$$j = NZev_e \int v_x(F-F_M)d\theta d\underset{\sim}{v} = j_n + j_r \, , \tag{92}$$

$$j_n = NZev_e \int_0^{2\pi} d\theta \int_n F_1 v_x d\underset{\sim}{v} \, , \tag{93}$$

$$j_r = NZev_e \int_0^{2\pi} d\theta \int_r (F-F_M) v_x d\underset{\sim}{v} \, , \tag{94}$$

where the subscripts n and r denote, respectively, the non-resonant region and the resonant region contributions and the integration limits are the same as in (65) and (66).

The non-resonant contribution j_n is the usual quasi-linear term which can be evaluated by solving F_1 in (22) and noting that only the part of F_1 that is anti-symmetric in v_x contributes. The resonant contribution j_r can be evaluated by substituting (34), (39) and (42) into (94). Using the symmetry in u, we obtain for j_r:

$$\begin{aligned} j_r &= NZv_e e \int d\theta \int_r d\underset{\sim}{v}(c+\epsilon u)(F-F_M) \\ &= \epsilon^2 NZv_e ec \int d\theta \int d\underset{\sim}{v}\left[c^2(H-u^2F_c/2T^2) + uG + u^2F_c/T\right] \end{aligned}$$

For large c, we only need the first two terms in the bracket of the last equation, and integrating by parts, we have

$$j_r \cong -4\pi\epsilon^3 c^3 NZv_e e \int d\theta v_\perp dv_\perp \int_{\xi_*}^{\Delta_*} d\xi |u| (H' - F_c/T^2)$$

which can be evaluated by using (42) and (48) and the final result is

$$j_r \cong \frac{1}{2}\pi(\epsilon^2 A/T)^{3/2} NZv_e e \ , \tag{95}$$

where β is the constant given in (85). We only note that the non-resonant current j_n is proportional to ϵ^4 while $j_r \sim \epsilon^3$.

VIII. CONCLUDING REMARKS

The main results of the present analysis show that the nonlinear decay term is proportional to $(\nu/\varepsilon)c^3 \exp(-c^2/2T)$ while the non-resonant linear term varies as ν. This indicates that the nonlinear term becomes important if the wave amplitude parameter ε is small. This must be understood in terms of the hypothesis that for the present analysis to be valid we must have $\varepsilon^3 \gg \nu$. When the amplitude becomes really small so that $\varepsilon^3 \ll \nu$, the resonant layer is governed by a collisional linear theory. In this case the resonant layer contribution leads to a linear Landau decay term that is independent of ν; here $\nu \ll 1$ but $\nu \gg \varepsilon^3$.

Similarly, we note that the resonant current is of order $\varepsilon^3 c^3 \exp(-c^2/2T)$, while the non-resonant current is of ε^4 order.

Because of the factor of $c^3 \exp(-c^2/2T)$, the resonant terms will be important if c is not too large, for instance, $c \lesssim 3T^{1/2}$. In particular, if $c \lesssim 2T^{1/2}$ and ε is small, the resonant terms will dominate and the wave evolution and the current will be primarily determined by the resonant layer contributions

$$\dot{A} \sim -\nu A^{1/2}/\varepsilon \ , \qquad j \sim \varepsilon^3 A^{3/2} \ . \tag{92}$$

For the case of $c \sim 1$, the analysis can be carried out with some modifications which mainly involve the evaluation of the plasma dispersion function and the treatment of the Fokker-Planck terms in the resonant layer without large c expansions. These modifications are not expected to change the qualitative feature of (92).

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